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The Maximal Riesz Operator of Two-Dimensional Fourier Transforms and Fourier Series on $H_p(\mathbf{R}\times\mathbf{R})$ and $H_p(\mathbf{T}\times\mathbf{T})$ [1]

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It is proved that the maximal operator of the two-parameter Riesz means with parameters $\alpha, \beta \leqslant 1$ is bounded from $L_p(\mathbf{R}^2)$ to $L_p(\mathbf{R}^2)$ $(1<p<\infty)$. The two-dimensional classical Hardy spaces $H_p(\mathbf{R}\times\mathbf{R})$ are introduced and it is shown that the maximal Riesz operator of a tempered distribution is also bounded from $H_p(\mathbf{R}\times\mathbf{R})$ to $L_p(\mathbf{R}^2)$ $(\max\{1/(\alpha+1),\ 1/(\beta+1)\}<p\leqslant\infty)$ and is of weak type $(H_1^\sharp(\mathbf{R}\times\mathbf{R}), L_1(\mathbf{R}^2))$ where the Hardy space $H_1^\sharp(\mathbf{R}\times\mathbf{R})$ is defined by the hybrid maximal function. As a consequence we obtain that the Riesz means of a function $f\in H_1^\sharp(\mathbf{R}\times\mathbf{R})\supset L\log L(\mathbf{R}^2)$ converge a.e. to the function in question. Moreover, we prove that the Riesz means are uniformly bounded on the spaces $H_p(\mathbf{R}\times\mathbf{R})$ whenever $\max\{1/(\alpha+1), 1/(\beta+1)\}<p<\infty$. Thus, in case $f\in H_p(\mathbf{R}\times\mathbf{R})$, the Riesz means converge to f in $H_p(\mathbf{R}\times\mathbf{R})$ norm. The same results are proved for the conjugate Riesz means and for two-parameter Fourier series, too.

Key Words: Hardy spaces; rectangle p-atom; atomic decomposition; interpolation; Riesz means.

1. INTRODUCTION

The Hardy–Lorentz spaces $H_{p,q}(\mathbf{R}\times\mathbf{R})$ of tempered distributions are introduced with the $L_{p,q}(\mathbf{R}^2)$ Lorentz norms of the non-tangential maximal function. Of course, $H_p(\mathbf{R}\times\mathbf{R})=H_{p,p}(\mathbf{R}\times\mathbf{R})$ are the usual Hardy spaces $(0<p\leqslant\infty)$.

In this paper the Riesz means $\sigma_{T,U}^{\alpha,\beta,\gamma,\delta}f$ of two-dimensional tempered distributions are considered where $0<\alpha, \beta<\infty$ and $1\leqslant\gamma, \delta<\infty$. In the one-dimensional case Butzer and Nessel [3] and Stein and Weiss [16] proved for $\gamma=1, 2$ that the Riesz means $\sigma_T^{\alpha,\gamma}f$ of a function $f\in L_1(\mathbf{R})$ converge a.e. to f as $T\to\infty$. The author [19] verified the same result for all $\gamma\geqslant 1$ and,

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moreover, that the one-dimensional maximal Riesz operator $\sigma_*^{\alpha,\gamma} := \sup_{T>0} |\sigma_T^{\alpha,\gamma}|$ is of weak type (1, 1), i.e.,

$$\sup_{\rho>0} \rho\lambda(\sigma_*^{\alpha,\gamma} f > \rho) \leqslant C \|f\|_1 \qquad (f \in L_1(\mathbf{R})).$$

Moreover, we proved in [19] that $\sigma_*^{\alpha,\gamma}$ is bounded from $H_p(\mathbf{R})$ to $L_p(\mathbf{R})$ provided that $1/(\alpha+1) < p < \infty$ and $0 < \alpha \leqslant 1$.

In Weisz [17] we investigated the Fejér means of two-parameter Fourier series, i.e. if $\alpha = \beta = \gamma = \delta = 1$, and proved that $\sigma_*^{1,1,1,1} := \sup_{n,m \in \mathbf{N}} |\sigma_{n,m}^{1,1,1,1}|$ is bounded from $H_{p,q}(\mathbf{T} \times \mathbf{T})$ to $L_{p,q}(\mathbf{T}^2)$ $(3/4 < p \leqslant \infty, 0 < q \leqslant \infty)$ and is of weak type $(H_1^\sharp(\mathbf{T} \times \mathbf{T}), L_1(\mathbf{T}^2))$, i.e.

$$\sup_{\rho>0} \rho\lambda(\sigma_*^{1,1,1,1} f > \rho) \leqslant C \|f\|_{H_1^\sharp(\mathbf{T}\times\mathbf{T})} \qquad (f \in H_1^\sharp(\mathbf{T} \times \mathbf{T})).$$

Moreover, the Fejér means $\sigma_{n,m}^{1,1,1,1} f$ converge a.e. to f as $n, m \to \infty$ whenever $f \in H_1^\sharp(\mathbf{T} \times \mathbf{T}) \supset L \log L(\mathbf{T}^2)$ (see Weisz [17] and Zygmund [21] for $L \log L(\mathbf{T}^2)$).

In this paper we use another method and so we can sharpen and generalize these results for the Riesz means of two-dimensional Fourier transforms and Fourier series with $\alpha, \beta > 0$ and $\gamma, \delta \geqslant 1$.

First we modify the one-dimensional Riesz means by taking the absolute value of the kernel functions and prove that the maximal operator of these modified Riesz means is of weak type $(L_1(\mathbf{R}), L_1(\mathbf{R}))$ and is bonded from $L_p(\mathbf{R})$ to $L_p(\mathbf{R})$ provided that $1 < p < \infty$. From this it follows that the maximal operator $\sigma_*^{\alpha,\gamma}$ of the original one-dimensional Riesz means is also of weak type $(L_1(\mathbf{R}), L_1(\mathbf{R}))$ and is bounded from $L_p(\mathbf{R})$ to $L_p(\mathbf{R})$ $(1 < p < \infty)$. Note that this last result was also proved in Weisz [19] with another method, as mentioned above. Using these two results about the one-dimensional Riesz means we verify that the two-dimensional maximal operator $\sigma_*^{\alpha,\beta,\gamma,\delta} := \sup_{T,U>0} |\sigma_{T,U}^{\alpha,\beta,\gamma,\delta}|$ $(0 < \alpha, \beta \leqslant 1)$ is bounded from $L_p(\mathbf{R}^2)$ to $L_p(\mathbf{R}^2)$ $(1 < p < \infty)$, which is also a new result.

Next we extend this result and investigate the boundedness of $\sigma_*^{\alpha,\beta,\gamma,\delta}$ on Hardy spaces. We will show that $\sigma_*^{\alpha,\beta,\gamma,\delta}$ is bounded from $H_{p,q}(\mathbf{R} \times \mathbf{R})$ to $L_{p,q}(\mathbf{R}^2)$ whenever $\max\{1/(\alpha+1), 1/(\beta+1)\} < p < \infty, 0 < q \leqslant \infty$ and is of weak type $(H_1^\sharp(\mathbf{R} \times \mathbf{R}), L_1(\mathbf{R}^2))$. We introduce the conjugate distributions $\tilde{f}^{(i,j)}$, the conjugate Riesz means $\tilde{\sigma}_{T,U}^{(i,j);\alpha,\beta,\gamma,\delta}$ and the conjugate maximal operators $\tilde{\sigma}_*^{(i,j);\alpha,\beta,\gamma,\delta}$ $(i, j = 0, 1)$. We obtain that the operator $\tilde{\sigma}_*^{(i,j);\alpha,\beta,\gamma,\delta}$ is also of type $(H_{p,q}(\mathbf{R} \times \mathbf{R}), L_{p,q}(\mathbf{R}^2))$ for $\max\{1/(\alpha+1), 1/(\beta+1)\} < p < \infty, 0 < q \leqslant \infty$ and of weak type $(H_1^\sharp(\mathbf{R} \times \mathbf{R}), L_1(\mathbf{R}^2))$.

A usual density argument implies then that the Riesz means $\sigma_{T,U}^{\alpha,\beta,\gamma,\delta} f$ converge a.e. to f and the conjugate Riesz means $\tilde{\sigma}_{T,U}^{(i,j);\alpha,\beta,\gamma,\delta} f$ converge a.e. to $\tilde{f}^{(i,j)}$ $(i, j = 0, 1)$ as $T, U \to \infty$, provided that $f \in H_1^\sharp(\mathbf{R} \times \mathbf{R})$. Note that $\tilde{f}^{(i,j)}$ is not necessarily in $H_1^\sharp(\mathbf{R} \times \mathbf{R})$ whenever f is.

We will prove also that the operators $\sigma_{T,U}^{\alpha,\beta,\gamma,\delta}$ and $\tilde{\sigma}_{T,U}^{(i,j);\alpha,\beta,\gamma,\delta}$ $(T, U \in \mathbf{R}_+)$ are uniformly bounded from $H_{p,q}(\mathbf{R}\times\mathbf{R})$ to $H_{p,q}(\mathbf{R}\times\mathbf{R})$ if $\max\{1/(\alpha+1), 1/(\beta+1)\} < p < \infty, 0 < q \leqslant \infty$. From this it follows that $\sigma_{T,U}^{\alpha,\beta,\gamma,\delta} f \to f$ and $\tilde{\sigma}_{T,U}^{(i,j);\alpha,\beta,\gamma,\delta} f \to \tilde{f}^{(i,j)}$ $(i, j = 0, 1)$ in $H_{p,q}(\mathbf{R}\times\mathbf{R})$ norm as $T, U \to \infty$, whenever $f \in H_{p,q}(\mathbf{R}\times\mathbf{R})$ and $\max\{1/(\alpha+1), 1/(\beta+1)\} < p < \infty$, $0 < q \leqslant \infty$.

We extend these results also for $\alpha > 1$ and/or $\beta > 1$.

We consider also the Riesz means of two-parameter Fourier series of distributions on $\mathbf{T}^2$ and prove all the results above in this context.

2. HARDY SPACES ON $\mathbf{R}\times\mathbf{R}$ AND CONJUGATE FUNCTIONS

For a set $\mathbf{X} \neq \emptyset$ let $\mathbf{X}^2$ be its Cartesian product $\mathbf{X}\times\mathbf{X}$ taken with itself, moreover, let $\mathbf{R}$ denote the real numbers, $\mathbf{R}_+$ the positive real numbers and let λ be the Lebesgue measure. We also use the notation $|I|$ for the Lebesgue measure of the set I. We briefly write $L_p(\mathbf{R}^2)$ instead of the real $L_p(\mathbf{R}^2, \lambda)$ space while the norm (or quasinorm) of this space is defined by $\|f\|_p := (\int_{\mathbf{R}^2} |f|^p \, d\lambda)^{1/p}$ $(0 < p \leqslant \infty)$.

The distribution function of a Lebesgue-measurable function f is defined by

$$\lambda(\{|f| > \rho\}) := \lambda(\{x : |f(x)| > \rho\}) \qquad (\rho \geqslant 0).$$

The *weak* $L_p(\mathbf{R}^2)$ space $L_p^*(\mathbf{R}^2)$ $(0 < p < \infty)$ consists of all measurable functions f for which

$$\|f\|_{L_p^*(\mathbf{R}^2)} := \sup_{\rho > 0} \rho \lambda(\{|f| > \rho\})^{1/p} < \infty$$

while we set $L_\infty^*(\mathbf{R}^2) = L_\infty(\mathbf{R}^2)$.

The spaces $L_p^*(\mathbf{R}^2)$ are special cases of the more general Lorentz spaces $L_{p,q}(\mathbf{R}^2)$. In their definition another concept is used. For a measurable function f the *non-increasing rearrangement* is defined by

$$\tilde{f}(t) := \inf\{\rho : \lambda(\{|f| > \rho\}) \leqslant t\}.$$

The Lorentz space $L_{p,q}(\mathbf{R}^2)$ is defined as follows: for $0 < p < \infty, 0 < q < \infty$

$$\|f\|_{p,q} := \left(\int_0^\infty \tilde{f}(t)^q \, t^{q/p} \frac{dt}{t}\right)^{1/q}$$

while for $0 < p \leqslant \infty$

$$\|f\|_{p,\infty} := \sup_{t > 0} t^{1/p} \tilde{f}(t).$$

Let

$$L_{p,q}(\mathbf{R}^2) := L_{p,q}(\mathbf{R}^2, \lambda) := \{f : \|f\|_{p,q} < \infty\}.$$

One can show the equalities

$$L_{p,p}(\mathbf{R}^2) = L_p(\mathbf{R}^2), \quad L_{p,\infty}(\mathbf{R}^2) = L_p^*(\mathbf{R}^2) \qquad (0 < p \leqslant \infty)$$

(see, e.g., Bennett and Sharpley [1] or Bergh and Löfström [2]).

Let f be a tempered distribution on $C^\infty(\mathbf{R}^2)$ (briefly $f \in \mathcal{S}'(\mathbf{R}^2)$). The *Fourier transform* of f is denoted by $\hat{f}$. In special case, if f is an integrable function then

$$\hat{f}(t, u) = \frac{1}{2\pi} \int_{\mathbf{R}} \int_{\mathbf{R}} f(x, y)\, e^{-\imath tx} e^{-\imath uy}\, dx\, dy \qquad (t, u \in \mathbf{R}),$$

where $\imath = \sqrt{-1}$.

For $f \in \mathcal{S}'(\mathbf{R}^2)$ and $t, u > 0$ let

$$F(x, y; t, u) := (f * P_t \otimes P_u)(x, y),$$

where $*$ denotes the convolution and

$$P_t(x) := \frac{ct}{t^2 + x^2} \qquad (x \in \mathbf{R})$$

is the Poisson kernel. Moreover, let $\Gamma := \{(x, t) : |x| < t\}$ a cone whose vertex is the origin. We denote by $\Gamma(x)$ $(x \in \mathbf{R})$ the translate of Γ so that its vertex is x. The non-tangential maximal function is defined by

$$F^*(x, y) := \sup_{(x', t) \in \Gamma(x),\, (y', u) \in \Gamma(y)} |F(x', y'; t, u)|.$$

For $0 < p, q \leqslant \infty$ the *Hardy–Lorentz space* $H_{p,q}(\mathbf{R} \times \mathbf{R})$ consists of all tempered distributions f for which $F^* \in L_{p,q}(\mathbf{R}^2)$ and set

$$\|f\|_{H_{p,q}(\mathbf{R} \times \mathbf{R})} := \|F^*\|_{p,q}.$$

It is known that if $f \in H_p(\mathbf{R} \times \mathbf{R})$ $(0 < p < \infty)$ then $f(x, y) = \lim_{t, u \to 0} F(x, y; t, u)$ in the sense of distributions (see Gundy and Stein [11], Chang and Fefferman [4]).

Let us introduce the hybrid Hardy spaces. For $f \in L_1(\mathbf{R}^2)$ and $t > 0$ let

$$G(x, y; t) := \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} f(v, y)\, P_t(x - v)\, dv$$

and

$$G^{+}(x, y) := \sup_{(x', t) \in \Gamma(x)} |G(x', y; t)|.$$

We say that $f \in L_1(\mathbf{R}^2)$ is in the *hybrid Hardy–Lorentz space* $H^{\sharp}_{p,q}(\mathbf{R}\times\mathbf{R})$ if

$$\|f\|_{H^{\sharp}_{p,q}(\mathbf{R}\times\mathbf{R})} := \|G^{+}\|_{p,q} < \infty.$$

The equivalences

$$H_{p,q}(\mathbf{R}\times\mathbf{R}) \sim H^{\sharp}_{p,q}(\mathbf{R}\times\mathbf{R}) \sim L_{p,q}(\mathbf{R}^2) \qquad (1<p<\infty,\ 0<q\leqslant\infty) \tag{1}$$

were proved in Fefferman and Stein [7], Gundy and Stein [11], and Lin [13]. Note that in case $p=q$ the usual definition of Hardy spaces $H_{p,p}(\mathbf{R}\times\mathbf{R}) = H_p(\mathbf{R}\times\mathbf{R})$ and $H^{\sharp}_{p,p}(\mathbf{R}\times\mathbf{R}) = H^{\sharp}_p(\mathbf{R}\times\mathbf{R})$ are obtained.

The following interpolation result concerning Hardy–Lorentz spaces will be used several times in this paper (see Lin [13] and also Weisz [18]).

THEOREM A. *If a sublinear* (*resp. linear*) *operator* V *is bounded from* $H_{p_0}(\mathbf{R}\times\mathbf{R})$ *to* $L_{p_0}(\mathbf{R}^2)$ (*resp. to* $H_{p_0}(\mathbf{R}\times\mathbf{R})$) *and from* $L_{p_1}(\mathbf{R}^2)$ *to* $L_{p_1}(\mathbf{R}^2)$ ($p_0 \leqslant 1 < p_1 < \infty$) *then it is also bounded from* $H_{p,q}(\mathbf{R}\times\mathbf{R})$ *to* $L_{p,q}(\mathbf{R}^2)$ (*resp. to* $H_{p,q}(\mathbf{R}\times\mathbf{R})$) *if* $p_0 < p < p_1$ *and* $0 < q \leqslant \infty$.

In this paper the constants C are absolute constants and the constants C_p (resp. $C_{p,q}$) are depending only on p (resp. p and q) and may denote different constants in different contexts.

One can prove similarly as in the discrete case (see Weisz [17]) that $L \log L(\mathbf{R}^2) \subset H^{\sharp}_1(\mathbf{R}\times\mathbf{R}) \subset H_{1,\infty}(\mathbf{R}\times\mathbf{R})$, more exactly,

$$\|f\|_{H_{1,\infty}(\mathbf{R}\times\mathbf{R})} = \sup_{\rho>0} \rho\lambda(F^{*}>\rho) \leqslant C\,\|f\|_{H^{\sharp}_1(\mathbf{R}\times\mathbf{R})} \qquad (f \in H^{\sharp}_1(\mathbf{R}\times\mathbf{R})) \tag{2}$$

and

$$\|f\|_{H^{\sharp}_1(\mathbf{R}\times\mathbf{R})} \leqslant C + C\,\||f| \log^{+}|f|\|_1 \qquad (f \in L \log L(\mathbf{R}^2)),$$

where $\log^{+} u = 1_{\{u>1\}} \log u$.

For a tempered distribution $f \in H_p(\mathbf{R}\times\mathbf{R})$ $(0<p<\infty)$ the *Hilbert transforms* or the *conjugate distributions* $\tilde{f}^{(1,0)}$, $\tilde{f}^{(0,1)}$ and $\tilde{f}^{(1,1)}$ are defined by

$$(\tilde{f}^{(1,0)})^{\wedge}(t, u) := (-\imath \operatorname{sign} t)\, \hat{f}(t, u) \qquad (t, u \in \mathbf{R})$$

(conjugate with respect to the first variable),

$$(\tilde{f}^{(0,1)})^{\wedge}(t, u) := (-\imath \operatorname{sign} u)\, \hat{f}(t, u) \qquad (t, u \in \mathbf{R})$$

(conjugate with respect to the second variable) and

$$(\tilde{f}^{(1,1)})^{\wedge}(t,u) := (-\operatorname{sign}(tu))\,\hat{f}(t,u) \qquad (t,u \in \mathbf{R})$$

(conjugate with respect to both variables), respectively. We use the notation $\tilde{f}^{(0,0)} := f$.

Gundy and Stein [10, 11] verified that if $f \in H_p(\mathbf{R}\times\mathbf{R})$ $(0<p<\infty)$ then all conjugate distributions are also in $H_p(\mathbf{R}\times\mathbf{R})$ and

$$\|f\|_{H_p(\mathbf{R}\times\mathbf{R})} = \|\tilde{f}^{(i,j)}\|_{H_p(\mathbf{R}\times\mathbf{R})} \qquad (i,j=0,1). \tag{3}$$

Furthermore (see also Chang and Fefferman [4], Frazier [9], Duren [5]),

$$\|f\|_{H_p(\mathbf{R}\times\mathbf{R})} \sim \|f\|_p + \|\tilde{f}^{(1,0)}\|_p + \|\tilde{f}^{(0,1)}\|_p + \|\tilde{f}^{(1,1)}\|_p. \tag{4}$$

As is well known, if f is an integrable function then

$$\tilde{f}^{(1,0)}(x,y) = \text{p.v.}\,\frac{1}{\pi}\int_{\mathbf{R}} \frac{f(x-t,y)}{t}\,dt := \lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{\varepsilon<|t|}\frac{f(x-t,y)}{t}\,dt,$$

$$\tilde{f}^{(0,1)}(x,y) = \text{p.v.}\,\frac{1}{\pi}\int_{\mathbf{R}} \frac{f(x,y-u)}{u}\,du,$$

and

$$\tilde{f}^{(1,1)}(x,y) = \text{p.v.}\,\frac{1}{\pi^2}\int_{\mathbf{R}}\int_{\mathbf{R}} \frac{f(x-t,y-u)}{tu}\,dt\,du.$$

Moreover, the conjugate function s $\tilde{f}^{(1,0)}$, $\tilde{f}^{(0,1)}$ and $\tilde{f}^{(1,1)}$ do exist almost everywhere, but they are not integrable in general. Similarly, if $f \in H_1^\sharp(\mathbf{R}\times\mathbf{R})$ then $\tilde{f}^{(0,1)}$ and $\tilde{f}^{(1,1)}$ are not necessarily in $H_1^\sharp(\mathbf{R}\times\mathbf{R})$.

3. THE TWO-DIMENSIONAL RIESZ MEANS OF FOURIER TRANSFORMS

Suppose first that $f \in L_p(\mathbf{R}^2)$ for some $1 \leqslant p \leqslant 2$. It is known that under certain conditions

$$f(x,y) = \frac{1}{2\pi}\int_{\mathbf{R}}\int_{\mathbf{R}} \hat{f}(t,u)\,e^{\imath xt}e^{\imath yu}\,dt\,du \qquad (x,y\in\mathbf{R}).$$

This motivates the definition of the *Dirichlet integral* $s_{t,u}f$,

$$s_{t,u}f(x,y):=\frac{1}{2\pi}\int_{-t}^{t}\int_{-u}^{u}\hat{f}(v,w)\,e^{\imath xv}e^{\imath yw}\,dv\,dw \qquad (t,u>0).$$

The *conjugate Dirichlet integrals* are introduced by

$$\tilde{s}_{t,u}^{(1,0)}f(x,y):=\frac{1}{2\pi}\int_{-t}^{t}\int_{-u}^{u}(-\imath\,\mathrm{sign}\,v)\,\hat{f}(v,w)\,e^{\imath xv}e^{\imath yw}\,dv\,dw \qquad (t,u>0),$$

$$\tilde{s}_{t,u}^{(0,1)}f(x,y):=\frac{1}{2\pi}\int_{-t}^{t}\int_{-u}^{u}(-\imath\,\mathrm{sign}\,w)\,\hat{f}(v,w)\,e^{\imath xv}e^{\imath yw}\,dv\,dw \qquad (t,u>0)$$

and

$$\tilde{s}_{t,u}^{(1,1)}f(x,y):=\frac{1}{2\pi}\int_{-t}^{t}\int_{-u}^{u}(-\ \mathrm{sign}\,(vw))\,\hat{f}(v,w)\,e^{\imath xv}e^{\imath yw}\,dv\,dw \qquad (t,u>0),$$

respectively. We write $s_{t,u}f=:\tilde{s}_{t,u}^{(0,0)}f$. It is easy to see that

$$\tilde{s}_{t,u}^{(i,j)}f(x,y):=\int_{\mathbf{R}}\int_{\mathbf{R}}\tilde{f}^{(i,j)}(x-v,y-w)\,\frac{\sin tv}{\pi v}\,\frac{\sin uw}{\pi w}\,dv\,dw \qquad (i,j=0,1).$$

For $\alpha,\beta,\gamma,\delta>0$ the *Riesz* and *conjugate Riesz means* are defined by

$$\tilde{\sigma}_{T,U}^{(i,j);\,\alpha,\beta,\gamma,\delta}f(x,y):=\frac{\alpha\beta\gamma\delta}{TU}\int_{0}^{T}\int_{0}^{U}\left(1-\left(\frac{t}{T}\right)^{\gamma}\right)^{\alpha-1}\left(\frac{t}{T}\right)^{\gamma-1}$$
$$\left(1-\left(\frac{u}{U}\right)^{\delta}\right)^{\beta-1}\left(\frac{u}{U}\right)^{\delta-1}\tilde{s}_{t,u}^{(i,j)}f(x,y)\,dt\,du,$$

where $T,U>0$ and $i,j=0,1$. Let $\sigma_{T,U}^{\alpha,\beta,\gamma,\delta}f:=\tilde{\sigma}_{T,U}^{(0,0);\,\alpha,\beta,\gamma,\delta}f$. Integrating by parts we get that

$$\sigma_{T,U}^{\alpha,\beta,\gamma,\delta}f(x,y)=\frac{1}{2\pi}\int_{\mathbf{R}}\int_{\mathbf{R}}f(x-t,y-u)\,K_T^{\alpha,\gamma}(t)\,K_U^{\beta,\delta}(u)\,dt\,du,$$

where

$$K_T^{\alpha,\gamma}(u):=\frac{2}{\sqrt{2\pi}}\int_{0}^{T}\left(1-\left(\frac{t}{T}\right)^{\gamma}\right)^{\alpha}\cos tu\,dt$$
$$=\frac{1}{\sqrt{2\pi}}\int_{-T}^{T}\left(1-\left|\frac{t}{T}\right|^{\gamma}\right)^{\alpha}\cos tu\,dt$$

is the Riesz kernel. Similarly,

$$\tilde{\sigma}_{T,U}^{(i,j);\alpha,\beta,\gamma,\delta} f(x,y) = \frac{1}{2\pi} \int_{\mathbf{R}} \int_{\mathbf{R}} \tilde{f}^{(i,j)}(x-t, y-u)\, K_T^{\alpha,\gamma}(t)\, K_U^{\beta,\delta}(u)\, dt\, du.$$

We verified in [19] that the Riesz kernel $K_T^{\alpha,\gamma}$ with $0 < \alpha \leqslant 1 \leqslant \gamma$ $(T > 0)$ satisfies the conditions

$$\int_{\mathbf{R}} |K_T^{\alpha,\gamma}|\, d\lambda \leqslant C, \tag{5}$$

$$|K_T^{\alpha,\gamma}(t)| \leqslant \frac{C}{T^\alpha\, |t|^{\alpha+1}} \qquad (t \in \mathbf{R},\ t \neq 0) \tag{6}$$

and

$$|(K_T^{\alpha,\gamma})'(t)| \leqslant \frac{C}{T^{\alpha-1}\, |t|^{\alpha+1}} \qquad (t \in \mathbf{R},\ t \neq 0), \tag{7}$$

where $(K_T^{\alpha,\gamma})'$ denotes the derivative of the Riesz kernel. Note that C may depend on α and γ.

The Riesz means are called typical means if $\gamma = \delta = 1$, Bochner–Riesz means if $\gamma = \delta = 2$ and Fejér means if $\alpha = \beta = \gamma = \delta = 1$. One can prove that (cf. Butzer and Nessel [3]),

$$\tilde{\sigma}_{T,U}^{(i,j);\alpha,\beta,\gamma,\delta} f(x,y) = \frac{1}{2\pi} \int_{-T}^{T} \int_{-U}^{U} \left(1 - \left|\frac{t}{T}\right|^\gamma\right)^\alpha \left(1 - \left|\frac{u}{U}\right|^\delta\right)^\beta \hat{f}(t,u)\, e^{\imath xt} e^{\imath yu}\, dt\, du,$$

$i, j = 0, 1$.

We extend the definition of the Riesz means to tempered distributions as

$$\sigma_{T,U}^{\alpha,\beta,\gamma,\delta} f := f * (K_T^{\alpha,\gamma} \times K_U^{\beta,\delta}) \qquad (T, U > 0).$$

One can show that $\sigma_{T,U}^{\alpha,\beta,\gamma,\delta} f$ is well defined for all tempered distributions $f \in H_p(\mathbf{R} \times \mathbf{R})$ $(0 < p \leqslant \infty)$ and for all functions $f \in L_p(\mathbf{R}^2)$ $(1 \leqslant p \leqslant \infty)$ (cf. Fefferman and Stein [7]). The extension of the conjugate Riesz means is

$$\tilde{\sigma}_{T,U}^{(i,j);\alpha,\beta,\gamma,\delta} f := \tilde{f}^{(i,j)} * (K_T^{\alpha,\gamma} \times K_U^{\beta,\delta}) \qquad (T, U > 0).$$

The *maximal* and *maximal conjugate Riesz operators* are defined by

$$\tilde{\sigma}_*^{(i,j);\alpha,\beta,\gamma,\delta} f := \sup_{T,U>0} |\tilde{\sigma}_{T,U}^{(i,j);\alpha,\beta,\gamma,\delta} f| \qquad (i, j = 0, 1).$$

We use again the notation $\sigma_*^{\alpha,\beta,\gamma,\delta} f := \tilde{\sigma}_*^{(0,0);\alpha,\beta,\gamma,\delta} f$.

4. THE BOUNDEDNESS OF THE MAXIMAL RIESZ OPERATOR ON $L_p(\mathbf{R}^2)$ SPACES

In order to prove that $\sigma_*^{\alpha,\beta,\gamma,\delta}$ is bounded on the $L_p(\mathbf{R}^2)$ $(1<p<\infty)$ spaces we consider first some one-parameter results. We modify the one-dimensional Riesz means by taking the absolute value of the kernel functions as follows. Let

$$\tau_T^{\alpha,\gamma} f(x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} f(x-u)\,|K_T^{\alpha,\gamma}(u)|\,du$$

and

$$\tau_*^{\alpha,\gamma} f := \sup_{T>0} |\tau_T^{\alpha,\gamma} f|.$$

Obviously,

$$|\sigma_T^{\alpha,\gamma} f| \leqslant \tau_T^{\alpha,\gamma}\,|f| \qquad (T>0).$$

With the help of the following theorem that was proved by Schipp *et al.* [15, pp. 262–263], we show that the operator $\tau_*^{\alpha,\gamma}$ is of weak type (1,1).

Theorem B. *Suppose that the sublinear operator V is bounded from $L_\infty(\mathbf{R})$ to $L_\infty(\mathbf{R})$ and*

$$\int_{\mathbf{R}\setminus 4I} |Vf|\,d\lambda \leqslant C\,\|f\|_1 \tag{8}$$

for all $f \in L_1(\mathbf{R})$ and intervals I which satisfy

$$\operatorname{supp} f \subset I \tag{9}$$

and

$$\int_{\mathbf{R}} f\,d\lambda = 0, \tag{10}$$

where rI $(r \in \mathbf{N})$ is the interval with the same center as I and with length $r\,|I|$. Then the operator V is of weak type (1,1), *i.e.*,

$$\lambda(|Vf|>\rho) \leqslant \frac{C}{\rho}\,\|f\|_1$$

for all $f \in L_1(\mathbf{R})$ and $\rho>0$.

Now we can formulate the main theorem of this section.

THEOREM 1. *Assume that* $0<\alpha\leqslant 1\leqslant\gamma$. *Then the operator* $\tau_*^{\alpha,\gamma}$ *is bounded from* $L_\infty(\mathbf{R})$ *to* $L_\infty(\mathbf{R})$ *and*

$$\lambda(\tau_*^{\alpha,\gamma} f>\rho)\leqslant\frac{C}{\rho}\,\|f\|_1$$

for all $f\in L_1(\mathbf{R})$ *and* $\rho>0$.

Proof. It is easy to see that (5) implies

$$\|\tau_*^{\alpha,\gamma} f\|_\infty\leqslant C\,\|f\|_\infty \qquad (f\in L_\infty(\mathbf{R})).$$

Let $f\in L_1(\mathbf{R})$ with support I which satisfy the conditions (9) and (10) and suppose that $2^{K-1}<|I|\leqslant 2^K$ $(K\in\mathbf{Z})$. We can suppose that the center of I is zero. In this case

$$[-2^{K-2},2^{K-2}]\subset I\subset[-2^{K-1},2^{K-1}].$$

Obviously,

$$\begin{aligned}\int_{\mathbf{R}\backslash 4I}\tau_*^{\alpha,\gamma} f(x)\,dx &\leqslant\sum_{|i|=1}^{\infty}\int_{i2^K}^{(i+1)\,2^K}\sup_{T\geqslant 2^{-K}}|\tau_T^{\alpha,\gamma} f(x)|\,dx\\ &\quad+\sum_{|i|=1}^{\infty}\int_{i2^K}^{(i+1)\,2^K}\sup_{T<2^{-K}}|\tau_T^{\alpha,\gamma} f(x)|\,dx\\ &=(A)+(B).\end{aligned}$$

We can suppose that $i\geqslant 1$.

Inequality (6) implies

$$|\tau_T^{\alpha,\gamma} f(x)|=\left|\int_I f(t)\,|K_T^{\alpha,\gamma}(x-t)|\,dt\right|\leqslant\int_I\frac{C\,|f(t)|}{T^\alpha\,|x-t|^{\alpha+1}}\,dt.$$

By a simple calculation we get

$$\frac{1}{|x-t|^{\alpha+1}}\leqslant\frac{C}{(i2^K-2^{K-1})^{\alpha+1}}\leqslant\frac{C\,2^{-K(\alpha+1)}}{i^{\alpha+1}} \tag{11}$$

if $x\in[i2^K,(i+1)\,2^K)$ $(i\geqslant 1)$. Hence

$$|\tau_T^{\alpha,\gamma} f(x)|\leqslant C\,2^{-K(\alpha+1)}T^{-\alpha}\frac{1}{i^{\alpha+1}}\int_I|f(t)|\,dt$$

and so

$$(A) \leqslant C \sum_{i=1}^{\infty} 2^{K} 2^{-K(\alpha+1)} 2^{K\alpha} \frac{1}{i^{(\alpha+1)}} \|f\|_1 = C \sum_{i=1}^{\infty} \frac{1}{i^{(\alpha+1)}} \|f\|_1 \leqslant C \|f\|_1. \qquad (12)$$

To estimate (B) observe that by (9) and (10)

$$\tau_T^{\alpha,\gamma} f(x) = \int_I f(t) \, |K_T^{\alpha,\gamma}(x-t)| \, dt = \int_I f(t)(|K_T^{\alpha,\gamma}(x-t)| - |K_T^{\alpha,\gamma}(x)|) \, dt.$$

Thus

$$|\tau_T^{\alpha,\gamma} f(x)| \leqslant \int_I |f(t)| \, |K_T^{\alpha,\gamma}(x-t) - K_T^{\alpha,\gamma}(x)| \, dt.$$

Using Lagrange's mean value theorem, (7) and (11) we conclude

$$\begin{aligned} |K_T^{\alpha,\gamma}(x-t) - K_T^{\alpha,\gamma}(x)| &= |(K_T^{\alpha,\gamma})' \, (x-\xi)| \, |t| \\ &\leqslant \frac{C\,|I|}{T^{\alpha-1} |x-\xi|^{\alpha+1}} \leqslant \frac{C\, 2^K 2^{-K(\alpha+1)} T^{1-\alpha}}{i^{\alpha+1}}, \end{aligned}$$

where $\xi \in I$ and $x \in [i2^K, (i+1)\, 2^K)$. Consequently,

$$|\tau_T^{\alpha,\gamma} f(x)| \leqslant C\, 2^{-K\alpha} T^{1-\alpha} \frac{1}{i^{\alpha+1}} \int_I |f(t)| \, dt$$

and

$$(B) \leqslant C \sum_{i=1}^{\infty} 2^{K} 2^{-K\alpha} 2^{-K(1-\alpha)} \frac{1}{i^{(\alpha+1)}} \|f\|_1 = C \sum_{i=1}^{\infty} \frac{1}{i^{(\alpha+1)}} \|f\|_1 \leqslant C \|f\|_1. \quad (13)$$

Theorem 1 follows now from (12), (13), and Theorem B. ∎

Note that for $\alpha = 1$ the proof is simpler because we do not need to estimate (A).

The following result follows by interpolation.

Theorem 2. *If* $0 < \alpha \leqslant 1 \leqslant \gamma$ *and* $1 < p \leqslant \infty$ *then*

$$\|\tau_*^{\alpha,\gamma} f\|_p \leqslant C_p \|f\|_p \qquad (f \in L_p(\mathbf{R})).$$

Since

$$\sigma_*^{\alpha,\gamma} f \leqslant \tau_*^{\alpha,\gamma} |f|,$$

we get

Corollary 1. *Assume that* $0<\alpha\leqslant 1\leqslant\gamma$ *and* $1<p\leqslant\infty$. *Then*

$$\lambda(\sigma_*^{\alpha,\gamma} f>\rho)\leqslant\frac{C}{\rho}\|f\|_1 \qquad (f\in L_1(\mathbf{R});\rho>0)$$

and

$$\|\sigma_*^{\alpha,\gamma} f\|_p\leqslant C_p\|f\|_p \qquad (f\in L_p(\mathbf{R})).$$

Note that Corollary 1 was proved also in Weisz [19] with another method.

Now we return to the two-dimensional case and verify the $L_p(\mathbf{R}^2)$ boundedness of $\sigma_*^{\alpha,\beta,\gamma,\delta}$.

Theorem 3. *Assume that* $0<\alpha,\beta\leqslant 1\leqslant\gamma,\delta$ *and* $1<p\leqslant\infty$. *Then*

$$\|\sigma_*^{\alpha,\beta,\gamma,\delta} f\|_p\leqslant C_p\|f\|_p \qquad (f\in L_p(\mathbf{R}^2)).$$

Proof. Applying Theorem 2 and Corollary 1 we have

$$\int_{\mathbf{R}}\int_{\mathbf{R}}\sup_{T,U\in\mathbf{R}_+}\left|\int_{\mathbf{R}}\int_{\mathbf{R}} f(t,u)\,K_T^{\alpha,\gamma}(x-t)\,K_U^{\beta,\delta}(y-u)\,dt\,du\right|^p dx\,dy$$

$$\leqslant\int_{\mathbf{R}}\int_{\mathbf{R}}\sup_{U\in\mathbf{R}_+}$$

$$\left[\int_{\mathbf{R}}\left(\sup_{T\in\mathbf{R}_+}\left|\int_{\mathbf{R}} f(t,u)\,K_T^{\alpha,\gamma}(x-t)\,dt\right|\right)|K_U^{\beta,\delta}(y-u)|\,du\right]^p dy\,dx$$

$$\leqslant C_p\int_{\mathbf{R}}\int_{\mathbf{R}}\sup_{T\in\mathbf{R}_+}\left|\int_{\mathbf{R}} f(t,y)\,K_T^{\alpha,\gamma}(x-t)\,dt\right|^p dx\,dy$$

$$\leqslant C_p\int_{\mathbf{R}}\int_{\mathbf{R}}|f(x,y)^p\,dx\,dy$$

which proves the theorem. ∎

Note that using the fact that $\sigma_*^{\alpha,\gamma}$ is bounded from $H_1(\mathbf{R})$ to $L_1(\mathbf{R})$ (see Weisz [19]) we could prove (15) in a similar way.

Since the set of those functions $f\in L_1(\mathbf{R}^2)$ whose Fourier transform has a compact support is dense in $L_p(\mathbf{R}^2)$ $(1<p<\infty)$ (see Wiener [20]), the usual density argument (see Marcinkievicz and Zygmund [14]) implies

COROLLARY 2. *Assume that* $0<\alpha, \beta \leqslant 1 \leqslant \gamma, \delta$ *and* $1<p<\infty$. *Then for every* $f \in L_p(\mathbf{R}^2)$ *we have*

$$\sigma_{T,U}^{\alpha,\beta,\gamma,\delta} f \to f \qquad \text{a.e. and in } L_p(\mathbf{R}^2) \text{ norm as} \qquad T, U \to \infty.$$

5. THE BOUNDEDNESS OF THE MAXIMAL RIESZ OPERATOR ON HARDY SPACES

In this section we consider the boundedness of $\sigma_*^{\alpha,\beta,\gamma,\delta}$ on the spaces $H_p(\mathbf{R}\times\mathbf{R})$ and extend Theorem 3 and Corollary 2.

A function $a \in L_2$ is called a *rectangle p-atom* if there exists a rectangle $R \subset \mathbf{R}^2$ such that

(i) $\operatorname{supp} a \subset R$

(ii) $\|a\|_2 \leqslant |R|^{1/2-1/p}$

(iii) for all $x, y \in \mathbf{R}$ and all $N \leqslant [2/p-3/2]$,

$$\int_{\mathbf{R}} a(x,y)\, x^N\, dx = \int_{\mathbf{R}} a(x,y)\, y^N\, dy = 0.$$

For a rectangle $R = I \times J$ let $rR = rI \times rJ$ $(r \in \mathbf{N})$. An operator V which maps the set of tempered distributions into the collection of measurable functions, will be called *p-quasi-local* if there exist a constant $C_p>0$ and $\eta>0$ such that for every rectangle p-atom a supported on the rectangle R and for every $r \geqslant 2$ one has

$$\int_{\mathbf{R}^2 \setminus 2^r R} |Ta|^p\, d\lambda \leqslant C_p 2^{-\eta r}.$$

Although $H_p(\mathbf{R}\times\mathbf{R})$ cannot be decomposed into rectangle p-atoms, in the next theorem it is enough to take these atoms (see Weisz [17], Fefferman [8]).

THEOREM C. *Suppose that the operator* V *is sublinear and p-quasi-local for some* $0<p\leqslant 1$. *If* V *is bounded from* $L_2(\mathbf{R}^2)$ *to* $L_2(\mathbf{R}^2)$ *then*

$$\|Vf\|_p \leqslant C_p \|f\|_{H_p(\mathbf{R}\times\mathbf{R})} \qquad (f \in H_p(\mathbf{R}\times\mathbf{R})).$$

Now we are in a position to state our main result.

THEOREM 4. *Assume that* $0<\alpha, \beta \leqslant 1 \leqslant \gamma, \delta$. *Then*

$$\|\sigma_*^{\alpha, \beta, \gamma, \delta} f\|_{p, q} \leqslant C_{p, q} \|f\|_{H_{p, q}(\mathbf{R}\times\mathbf{R})} \qquad (f \in H_{p, q}(\mathbf{R}\times\mathbf{R})) \tag{14}$$

for every $\max\{1/(\alpha+1),\ 1/(\beta+1)\}<p<\infty$ *and* $0<q\leqslant\infty$. *Especially, if* $f \in H_1^\sharp(\mathbf{R}\times\mathbf{R})$ *then*

$$\lambda(\sigma_*^{\alpha, \beta, \gamma, \delta} f>\rho) \leqslant \frac{C}{\rho} \|f\|_{H_1^\sharp(\mathbf{R}\times\mathbf{R})} \qquad (\rho>0). \tag{15}$$

Proof. First we will show that the operator $\sigma_*^{\alpha, \beta, \gamma, \delta}$ is p-quasi-local for each $\max\{1/(\alpha+1),\ 1/(\beta+1)\}<p\leqslant 1$. To this end let a be an arbitrary rectangle p-atom with support $R=I\times J$ and

$$2^{K-1}<|I|\leqslant 2^K, \quad 2^{L-1}<|J|\leqslant 2^L \qquad (K, L \in \mathbf{Z}).$$

We can suppose again that

$$[-2^{K-2}, 2^{K-2}] \subset I \subset [-2^{K-1}, 2^{K-1}]$$

and

$$[-2^{L-2}, 2^{L-2}] \subset J \subset [-2^{L-1}, 2^{L-1}].$$

To prove the p-quasi-locality of the operator $\sigma_*^{\alpha, \beta, \gamma, \delta}$ we have to integrate $|\sigma_*^{\alpha, \beta, \gamma, \delta} a|^p$ over

$$\mathbf{R}^2\backslash 2^r R=(\mathbf{R}\backslash 2^r I)\times J \bigcup (\mathbf{R}\backslash 2^r I)\times(\mathbf{R}\backslash J)$$

$$\bigcup I\times(\mathbf{R}\backslash 2^r J) \bigcup (\mathbf{R}\backslash I)\times(\mathbf{R}\backslash 2^r J),$$

where $r\geqslant 2$ is an arbitrary integer.

First we integrate over $(\mathbf{R}\backslash 2^r I)\times J$. We have

$$\begin{aligned} &\int_{\mathbf{R}\backslash 2^r I}\int_J |\sigma_*^{\alpha, \beta, \gamma, \delta} a(x, y)|^p\, dx\, dy \\ &\quad\leqslant \sum_{|i|=2^{r-2}}^{\infty} \int_{i2^K}^{(i+1)\,2^K}\int_J \sup_{T<2^{-K},\, U\in\mathbf{R}_+} |\sigma_{T, U}^{\alpha, \beta, \gamma, \delta} a(x, y)|^p\, dx\, dy \\ &\qquad+\sum_{|i|=2^{r-2}}^{\infty} \int_{i2^K}^{(i+1)\,2^K}\int_J \sup_{T\geqslant 2^{-K},\, U\in\mathbf{R}_+} |\sigma_{T, U}^{\alpha, \beta, \gamma, \delta} a(x, y)|^p\, dx\, dy \\ &\quad=(A)+(B). \end{aligned} \tag{16}$$

For $x, y \in \mathbf{R}$ let

$$A_{1,0}(x, y) := \int_{-\infty}^{x} a(t, y)\, dt, \qquad A_{0,1}(x, y) := \int_{-\infty}^{y} a(x, u)\, du$$

and

$$A_{1,1}(x, y) := \int_{-\infty}^{x} \int_{-\infty}^{y} a(t, y)\, dt\, du.$$

By (iii) of the definition of the rectangle atom we can show that supp $A_{k,l} \subset R$ and $A_{k,l}$ is zero at the limit points of R $(k, l = 0, 1)$. Moreover, using (ii) we can compute that

$$\|A_{k,l}\|_2 \leqslant |I|^k\, |J|^l (|I|\, |J|)^{1/2 - 1/p} \qquad (k, l = 0, 1). \tag{17}$$

Integrating by parts we can see that

$$|\sigma_{T,U}^{\alpha,\beta,\gamma,\delta} a(x, y)| = \left| \int_I \int_J A_{1,0}(t, u) (K_T^{\alpha,\gamma})'\,(x - t)\, K_U^{\beta,\delta}(y - u)\, dt\, du \right|$$

$$\leqslant \int_I \left| \int_J A_{1,0}(t, u)\, K_U^{\beta,\delta}(y - u)\, du \right| |(K_T^{\alpha,\gamma})'\,(x - t)|\, dt.$$

Apply (7) and (11) to obtain

$$|\sigma_{T,U}^{\alpha,\beta,\gamma,\delta} a(x, y)| \leqslant \int_I \left| \int_J A_{1,0}(t, u)\, K_U^{\beta,\delta}(y - u)\, du \right| \frac{C}{T^{\alpha-1}\, |x - t|^{\alpha+1}}\, dt$$

$$\leqslant \frac{C\, 2^{-K(\alpha+1)} T^{1-\alpha}}{i^{\alpha+1}} \int_I \left| \int_J A_{1,0}(t, u)\, K_U^{\beta,\delta}(y - u)\, du \right| dt$$

provided that $x \in [i2^K, (i+1)\, 2^K)$ $(i \geqslant 1)$. Hölder's inequality implies

$$\int_J \sup_{T < 2^{-K},\, U \in \mathbf{R}_+} |\sigma_{T,U}^{\alpha,\beta,\gamma,\delta} a(x, y)|^p\, dy$$

$$\leqslant \frac{C_p 2^{-K(\alpha+1)p}\, 2^{-K(1-\alpha)p}}{i^{(\alpha+1)p}} |J|^{1-p}$$

$$\left(\int_I \int_J \sup_{U \in \mathbf{R}_+} \left| \int_J A_{1,0}(t, u)\, K_U^{\beta,\delta}(y - u)\, du \right| dy\, dt \right)^p. \tag{18}$$

Using again Hölder's inequality, Corollary 1, and (17) we can conclude that

$$\int_I \int_J \sup_{U \in \mathbf{R}_+} \left| \int_J A_{1,0}(t,u)\, K_U^{\beta,\delta}(y-u)\, du \right| dy\, dt \tag{19}$$

$$\leqslant |J|^{1/2} \int_I \left(\int_{\mathbf{R}} \sup_{U \in \mathbf{R}_+} \left| \int_J A_{1,0}(t,u)\, K_U^{\beta,\gamma}(y-u)\, du \right|^2 dy \right)^{1/2} dt$$

$$\leqslant C\, |J|^{1/2} \int_I \left(\int_J |A_{1,0}(t,y)|^2\, dy \right)^{1/2} dt$$

$$\leqslant C\, |I|^{1/2}\, |J|^{1/2} \left(\int_I \int_J |A_{1,0}(t,y)|^2\, dy\, dt \right)^{1/2}$$

$$\leqslant C\, |I|^{2-1/p}\, |J|^{1-1/p}.$$

Consequently,

$$(A) \leqslant C_p \sum_{i=2^r-2}^{\infty} 2^K \frac{2^{-2Kp}}{i^{(\alpha+1)p}}\, 2^{L-Lp}\, 2^{2Kp-K}\, 2^{Lp-L} \tag{20}$$

$$\leqslant C_p \sum_{i=2^r-2}^{\infty} \frac{1}{i^{(\alpha+1)p}} \leqslant C_p 2^{-r((\alpha+1)p-1)}.$$

Similarly, we get by (6) that

$$|\sigma_{T,U}^{\alpha,\beta,\gamma,\delta} a(x,y)| \leqslant \int_I \left| \int_J a(t,u)\, K_U^{\beta,\delta}(y-u)\, du \right| |K_T^{\alpha,\gamma}(x-t)|\, dt$$

$$\leqslant \frac{C 2^{-K(\alpha+1)} T^{-\alpha}}{i^{\alpha+1}} \int_I \left| \int_J a(t,u)\, K_U^{\beta,\delta}(y-u)\, du \right| dt$$

whenever $x \in [i\, 2^K, (i+1)\, 2^K)$. Then

$$\int_J \sup_{T \geqslant 2^{-K},\, U \in \mathbf{R}_+} |\sigma_{T,U}^{\alpha,\beta,\gamma,\delta} a(x,y)|^p\, dy$$

$$\leqslant \frac{C_p 2^{-K(\alpha+1)p}\, 2^{K\alpha p}}{i^{(\alpha+1)p}} |J|^{1-p} \left(\int_I \int_J \sup_{U \in \mathbf{R}_+} \left| \int_J a(t,u)\, K_U^{\beta,\gamma}(y-u)\, du \right| dy\, dt \right)^p.$$

Applying an analogous inequality to (19) we can establish that

$$\begin{aligned}(B) &\leqslant C_p \sum_{i=2^r-2}^{\infty} 2^K \frac{2^{-Kp}}{i^{(\alpha+1)p}}\, 2^{L-Lp}\, 2^{Kp-K}\, 2^{Lp-L} \\ &\leqslant C_p \sum_{i=2^r-2}^{\infty} \frac{1}{i^{(\alpha+1)p}} \leqslant C_p 2^{-r((\alpha+1)p-1)}.\end{aligned} \tag{21}$$

Inequalities (20) and (21) imply

$$\int_{\mathbf{R}\backslash 2^r I} \int_J |\sigma_*^{\alpha, \beta, \gamma, \delta} a(x, y)|^p \, dx \, dy \leqslant C_p 2^{-\eta r}. \tag{22}$$

Next we integrate over $(\mathbf{R}\backslash 2^r I) \times (\mathbf{R}\backslash J)$. Similarly to (16),

$$\begin{aligned}
&\int_{\mathbf{R}\backslash 2^r I} \int_{\mathbf{R}\backslash J} |\sigma_*^{\alpha, \beta, \gamma, \delta} a(x, y)|^p \, dx \, dy \\
&\quad \leqslant \sum_{|i| = 2^{r-2}}^{\infty} \sum_{|j|=1}^{\infty} \int_{i2^K}^{(i+1)\, 2^K} \int_{j2^L}^{(j+1)\, 2^L} \sup_{T < 2^{-K},\, U < 2^{-L}} |\sigma_{T, U}^{\alpha, \beta, \gamma, \delta} a(x, y)|^p \, dx \, dy \\
&\qquad + \sum_{|i| = 2^{r-2}}^{\infty} \sum_{|j|=1}^{\infty} \int_{i2^K}^{(i+1)\, 2^K} \int_{j2^L}^{(j+1)\, 2^L} \sup_{T < 2^{-K},\, U \geqslant 2^{-L}} |\sigma_{T, U}^{\alpha, \beta, \gamma, \delta} a(x, y)|^p \, dx \, dy \\
&\qquad + \sum_{|i| = 2^{r-2}}^{\infty} \sum_{|j|=1}^{\infty} \int_{i2^K}^{(i+1)\, 2^K} \int_{j2^L}^{(j+1)\, 2^L} \sup_{T \geqslant 2^{-K},\, U < 2^{-L}} |\sigma_{T, U}^{\alpha, \beta, \gamma, \delta} a(x, y)|^p \, dx \, dy \\
&\qquad + \sum_{|i| = 2^{r-2}}^{\infty} \sum_{|j|=1}^{\infty} \int_{i2^K}^{(i+1)\, 2^K} \int_{j2^L}^{(j+1)\, 2^L} \sup_{T \geqslant 2^{-K},\, U \geqslant 2^{-L}} |\sigma_{T, U}^{\alpha, \beta, \gamma, \delta} a(x, y)|^p \, dx \, dy \\
&\quad = (C) + (D) + (E) + (F).
\end{aligned}$$

Integrating by parts and using (7), (11), and (17) we conclude that

$$\begin{aligned}
|\sigma_{T, U}^{\alpha, \beta, \gamma, \delta} a(x, y)| &= \left| \int_I \int_J A_{1, 1}(t, u) (K_T^{\alpha, \gamma})' \, (x - t) (K_U^{\beta, \delta})' \, (y - u) \, dt \, du \right| \qquad (23) \\
&\leqslant \int_I \int_J |A_{1, 1}(t, u)| \frac{C}{T^{\alpha - 1} \, |x - t|^{\alpha + 1}} \frac{C}{U^{\beta - 1} \, |y - u|^{\beta + 1}} \, dt \, du \\
&\leqslant \frac{C 2^{-K(\alpha + 1)} 2^{-L(\beta + 1)} T^{1 - \alpha} U^{1 - \beta}}{i^{\alpha + 1} j^{\beta + 1}} \int_I \int_J |A_{1, 1}(t, u)| \, dt \, du \\
&\leqslant \frac{C 2^{-K(\alpha + 1)} 2^{-L(\beta + 1)} T^{1 - \alpha} U^{1 - \beta}}{i^{\alpha + 1} j^{\beta + 1}} |I|^{1/2} \, |J|^{1/2} \, \|A_{1, 1}\|_2 \\
&\leqslant \frac{C 2^{-K\alpha + K - K/p} 2^{-L\beta + L - L/p} T^{1 - \alpha} U^{1 - \beta}}{i^{\alpha + 1} j^{\beta + 1}},
\end{aligned}$$

where $x \in [\, i2^K, (i+1)\, 2^K)$, $y \in [\, j2^L, (j+1)\, 2^L)$. Henceforth

$$(C) \leqslant C_p \sum_{i = 2^{r-2}}^{\infty} \sum_{j=1}^{\infty} 2^{K+L} \frac{2^{-K} 2^{-L}}{i^{(\alpha + 1)\, p} j^{(\beta + 1)\, p}} \leqslant C_p 2^{-r((\alpha + 1)\, p - 1)}. \tag{24}$$

We get in the same way as in (23) that

$$|\sigma_{T,U}^{\alpha,\beta,\gamma,\delta} a(x,y)| = \left|\int_I \int_J A_{1,0}(t,u)(K_T^{\alpha,\gamma})'(x-t)\, K_U^{\beta,\delta}(y-u)\, dt\, du\right|$$
$$\leqslant \frac{C 2^{-K(\alpha+1)} 2^{-L(\beta+1)} T^{1-\alpha} U^{-\beta}}{i^{\alpha+1} j^{\beta+1}} \int_I \int_J |A_{1,0}(t,u)|\, dt\, du$$
$$\leqslant \frac{C 2^{-K\alpha+K-K/p} 2^{-L\beta-L/p} T^{1-\alpha} U^{-\beta}}{i^{\alpha+1} j^{\beta+1}}$$

in case $x \in [i2^K, (i+1)\, 2^K)$, $y \in [j2^L, (j+1)\, 2^L)$. Hence the inequality (24) is also true for (D). The estimation of (E) is similar.

To estimate (F) let us observe that

$$|\sigma_{T,U}^{\alpha,\beta,\gamma,\delta} a(x,y)| = \left|\int_I \int_J a(t,u)\, K_T^{\alpha,\gamma}(x-t)\, K_U^{\beta,\delta}(y-u)\, dt\, du\right|$$
$$\leqslant \frac{C 2^{-K(\alpha+1)} 2^{-L(\beta+1)} T^{-\alpha} U^{-\beta}}{i^{\alpha+1} j^{\beta+1}} \int_I \int_J |a(t,u)|\, dt\, du$$
$$\leqslant \frac{C 2^{-K\alpha-K/p} 2^{-L\beta-L/p} T^{-\alpha} U^{-\beta}}{i^{\alpha+1} j^{\beta+1}}$$

provided that $x \in [i2^K, (i+1)\, 2^K)$, $y \in [j2^L, (j+1)\, 2^L)$. This implies that (F) satisfies also (24).

Consequently,

$$\int_{\mathbf{R}\backslash 2^r I} \int_{\mathbf{R}\backslash J} |\sigma_*^{\alpha,\beta,\gamma,\delta} a(x,y)|^p\, dx\, dy \leqslant C_p 2^{-\eta r}. \tag{25}$$

The integrations over $I \times (\mathbf{R}\backslash 2^r J)$ and over $(\mathbf{R}\backslash I) \times (\mathbf{R}\backslash 2^r J)$ are similar. Hence (22) and (25) imply the p-quasi-locality of $\sigma_*^{\alpha,\beta,\gamma,\delta}$.

Inequality (14) for $\max\{1/(\alpha+1),\ 1/(\beta+1)\} < p = q \leqslant 1$ follows now from Theorems 3 and C. Applying Theorems A and 3 we obtain (14).

Let us point out this result for $p=1$ and $q=\infty$. If $f \in H_1^\sharp(\mathbf{R}\times\mathbf{R})$ then (2) implies

$$\|\sigma_*^{\alpha,\beta,\gamma,\delta} f\|_{1,\infty} = \sup_{\rho>0} \rho\lambda(\sigma_*^{\alpha,\beta,\gamma,\delta} f > \rho) \leqslant C\, \|f\|_{H_{1,\infty}(\mathbf{R}\times\mathbf{R})} \leqslant C\, \|f\|_{H_1^\sharp(\mathbf{R}\times\mathbf{R})}$$

which shows (15). The proof of the theorem is complete. ∎

Remark. The proof of Theorem 4 is simpler if $\alpha = \beta = 1$ because we do not need the estimations of (B), (D), (E), and (F).

We can state the same for the maximal conjugate Riesz operators.

THEOREM 5. *Assume that* $0<\alpha, \beta \leqslant 1 \leqslant \gamma, \delta$ *and* $i, j=0, 1$. *Then*

$$\|\tilde{\sigma}_*^{(i,j);\alpha,\beta,\gamma,\delta} f\|_{p,q} \leqslant C_{p,q} \|f\|_{H_{p,q}(\mathbf{R}\times\mathbf{R})} \qquad (f \in H_{p,q}(\mathbf{R}\times\mathbf{R}))$$

for every $\max\{1/(\alpha+1), 1/(\beta+1)\} < p < \infty$ *and* $0 < q \leqslant \infty$. *Especially, if* $f \in H_1^\sharp(\mathbf{R}\times\mathbf{R})$ *then*

$$\lambda(\tilde{\sigma}_*^{(i,j);\alpha,\beta,\gamma,\delta} f > \rho) \leqslant \frac{C}{\rho} \|f\|_{H_1^\sharp(\mathbf{R}\times\mathbf{R})} \qquad (\rho > 0).$$

Proof. By Theorem 4 for $p=q$ and (3) we obtain

$$\|\tilde{\sigma}_*^{(i,j);\alpha,\beta,\gamma,\delta} f\|_p = \|\sigma_*^{\alpha,\beta,\gamma,\delta} \tilde{f}^{(i,j)}\|_p \leqslant C_p \|\tilde{f}^{(i,j)}\|_{H_p(\mathbf{R}\times\mathbf{R})} = C_p \|f\|_{H_p(\mathbf{R}\times\mathbf{R})}$$

for every $\max\{1/(\alpha+1), 1/(\beta+1)\} < p < \infty$. Now Theorem 5 follows from Theorem A and from (2). ∎

Since the set of those functions $f \in L_1(\mathbf{R}^2)$ whose Fourier transform has a compact support is dense in $H_1^\sharp(\mathbf{R}\times\mathbf{R})$, the weak type inequalities of Theorems 4 and 5 imply

COROLLARY 3. *Assume that* $0<\alpha, \beta \leqslant 1 \leqslant \gamma, \delta$ *and* $i, j=0, 1$. *If* $f \in H_1^\sharp(\mathbf{R}\times\mathbf{R})$ $(\supset L \log L(\mathbf{R}^2))$ *then*

$$\tilde{\sigma}_{T,U}^{(i,j);\alpha,\beta,\gamma,\delta} f \to \tilde{f}^{(i,j)} \qquad \textit{a.e.} \quad \textit{as} \quad T, U \to \infty.$$

Note that $\tilde{f}^{(i,j)}$ is not necessarily in $H_1^\sharp(\mathbf{R}\times\mathbf{R})$ whenever f is.

Now we consider the norm convergence of $\sigma_{T,U}^{\alpha,\beta,\gamma,\delta} f$ and extend Corollary 2.

THEOREM 6. *Assume that* $0<\alpha, \beta \leqslant 1 \leqslant \gamma, \delta, i, j=0, 1$ *and* $T, U \in \mathbf{R}_+$. *Then*

$$\|\tilde{\sigma}_{T,U}^{(i,j);\alpha,\beta,\gamma,\delta} f\|_{H_{p,q}(\mathbf{R}\times\mathbf{R})} \leqslant C_{p,q} \|f\|_{H_{p,q}(\mathbf{R}\times\mathbf{R})} \qquad (f \in H_{p,q}(\mathbf{R}\times\mathbf{R}))$$

for every $\max\{1/(\alpha+1), 1/(\beta+1)\} < p < \infty$ *and* $0 < q \leqslant \infty$.

Proof. Since $(\sigma_{T,U}^{\alpha,\beta,\gamma,\delta} f)^{\sim(i,j)} = \tilde{\sigma}_{T,U}^{(i,j);\alpha,\beta,\gamma,\delta} f$, we have by Theorem 5 that

$$\|(\sigma_{T,U}^{\alpha,\beta,\gamma,\delta} f)^{\sim(i,j)}\|_p \leqslant C_p \|f\|_{H_p(\mathbf{R}\times\mathbf{R})} \qquad (f \in H_p(\mathbf{R}\times\mathbf{R}))$$

for all $T, U \in \mathbf{R}_+$ and $i, j=0, 1$. The inequality

$$\|\sigma_{T,U}^{\alpha,\beta,\gamma,\delta} f\|_{H_p(\mathbf{R}\times\mathbf{R})} \leqslant C_p \|f\|_{H_p(\mathbf{R}\times\mathbf{R})} \qquad (f \in H_p(\mathbf{R}\times\mathbf{R}); T, U \in \mathbf{R}_+)$$

follows from (4). Hence, for $i, j=0, 1$,

$$\|\tilde{\sigma}_{T,U}^{(i,j);\alpha,\beta,\gamma,\delta} f\|_{H_p(\mathbf{R}\times\mathbf{R})} \leqslant C_p \|f\|_{H_p(\mathbf{R}\times\mathbf{R})} \qquad (f\in H_p(\mathbf{R}\times\mathbf{R});\ T, U\in\mathbf{N}).$$

Now Theorem 6 follows from Theorem A. ∎

COROLLARY 4. *Assume that* $0<\alpha, \beta\leqslant 1\leqslant \gamma, \delta$ *and* $i, j=0, 1$. *If* $\max\{1/(\alpha+1), 1/(\beta+1)\}<p<\infty$, $0<q\leqslant\infty$ *and* $f\in H_{p,q}(\mathbf{R}\times\mathbf{R})$ *then*

$$\tilde{\sigma}_{T,U}^{(i,j);\alpha,\beta,\gamma,\delta} f\to \tilde{f}^{(i,j)} \qquad \textit{in } H_{p,q}(\mathbf{R}\times\mathbf{R}) \textit{ norm} \quad \textit{as} \quad T, U\to\infty.$$

We suspect that Theorems 4, 5, and 6 for $p\leqslant \max\{1/(\alpha+1), 1/(\beta+1)\}$ are not true though we could not find any counterexample.

We will extend the results to $\alpha>1$ and $\beta>1$. By integrating by parts we proved in [19] that

$$\sigma_{T,U}^{1+h,\beta,\gamma,\delta} f(x,y)=\frac{h(h+1)\gamma}{T}\int_0^T \left(1-\left(\frac{s}{T}\right)^{\gamma}\right)^{h-1} \left(\frac{s}{T}\right)^{2\gamma-1} \sigma_{s,U}^{1,\beta,\gamma,\delta} f(x,y)\,ds,$$

where $h>0$. In other words

$$\sigma_*^{\alpha,\beta,\gamma,\delta} f\leqslant C\sigma_*^{\alpha\wedge 1,\beta\wedge 1,\gamma,\delta} f$$

which shows that Theorems 3–5 hold also for $\alpha>1$ and/or $\beta>1$. The extension of Theorem 6 can be proved in the same way.

COROLLARY 5. *If* $0<\alpha, \beta<\infty$ *and* $1\leqslant\gamma, \delta$ *then all inequalities of Theorems* 3–6 *and all convergence results of Corollaries* 2–4 *hold for every* $\max\{1/(1+\alpha\wedge 1), 1/(1+\beta\wedge 1)\}<p<\infty$ *and* $0<q\leqslant\infty$.

In the next sections we verify the results above in the periodic case, i.e., for the Riesz summability of two-parameter Fourier series.

6. HARDY SPACES ON THE BIDISC AND CONJUGATE FUNCTIONS

The Lorentz spaces on the measure space $(\mathbf{T}^2:=[-\pi,\pi)^2,\lambda)$ are denoted by $L_{p,q}(\mathbf{T}^2)$. Let f be a distribution on $C^\infty(\mathbf{T}^2)$ (briefly $f\in\mathcal{D}'(\mathbf{T}^2)$). The (n,m)th Fourier coefficient is defined by $\hat{f}(n,m):=f(e^{-\imath nx}e^{-\imath my})$. In special case, if f is an integrable function then

$$\hat{f}(n,m)=\frac{1}{(2\pi)^2}\int_{\mathbf{T}}\int_{\mathbf{T}} f(x,y)\,e^{-\imath nx}e^{-\imath my}\,dx\,dy.$$

For simplicity, we assume that, for a distribution $f \in \mathcal{D}'(\mathbf{T}^2)$, we have $\hat{f}(n, 0) = \hat{f}(0, n) = 0$ $(n \in \mathbf{N})$.

For $f \in \mathcal{D}'(\mathbf{T}^2)$ and $z_1 := re^{\imath x}$, $z_2 := se^{\imath y}$ $(0 < r, s < 1)$ let

$$u(z_1, z_2) = u(re^{\imath x}, se^{\imath y}) := (f * P_r \otimes P_s)(x, y),$$

where

$$P_r(x) := \sum_{k=-\infty}^{\infty} r^{|k|} e^{\imath kx} = \frac{1 - r^2}{1 + r^2 - 2r \cos x} \qquad (x \in \mathbf{T})$$

is the periodic Poisson kernel.

The non-tangential maximal function is defined by

$$u^*(x, y) := \sup_{z_1 \in \Omega(x)} \sup_{z_2 \in \Omega(y)} |u(z_1, z_2)|,$$

where $\Omega(x)$ is the usual Stolz domain (see, e.g., Kashin and Saakyan [12] or Weisz [17]).

For $0 < p, q \leqslant \infty$ the *Hardy–Lorentz space* $H_{p,q}(\mathbf{T} \times \mathbf{T})$ consists of all distributions f for which $u^* \in L_{p,q}(\mathbf{T}^2)$ and set

$$\|f\|_{H_{p,q}(\mathbf{T} \times \mathbf{T})} := \|u^*\|_{p,q}.$$

For $f \in L_1(\mathbf{T}^2)$ and $z := re^{\imath x}$ $(0 < r < 1)$ let

$$v(z, y) = v(re^{\imath x}, y) := \frac{1}{2\pi} \int_{\mathbf{T}} f(t, y)\, P_r(x - t)\, dt$$

and

$$v^+(x, y) := \sup_{z \in \Omega(x)} |v(z, y)|.$$

We say that $f \in L_1(\mathbf{T}^2)$ is in the *hybrid Hardy–Lorentz space* $H^\sharp_{p,q}(\mathbf{T} \times \mathbf{T})$ if

$$\|f\|_{H^\sharp_{p,q}(\mathbf{T} \times \mathbf{T})} := \|v^+\|_{p,q} < \infty.$$

For a distribution

$$f \sim \sum_{k, l \in \mathbf{Z}} \hat{f}(k, l)\, e^{\imath kx + \imath ly}$$

the *conjugate distributions* are defined by

$$\tilde{f}^{(1,0)} \sim \sum_{k,l \in \mathbf{Z}} (-\imath \operatorname{sign} k)\, \hat{f}(k,l)\, e^{\imath kx + \imath ly},$$

$$\tilde{f}^{(0,1)} \sim \sum_{k,l \in \mathbf{Z}} (-\imath \operatorname{sign} l)\, \hat{f}(k,l)\, e^{\imath kx + \imath ly}$$

and

$$\tilde{f}^{(1,1)} \sim \sum_{k,l \in \mathbf{Z}} (-\operatorname{sign}(kl))\, \hat{f}(k,l)\, e^{\imath kx + \imath ly},$$

respectively. We use again the notation $\tilde{f}^{(0,0)} := f$. If f is an integrable function then

$$\tilde{f}^{(1,0)}(x,y) = \text{p.v. } \frac{1}{\pi} \int_{\mathbf{T}} \frac{f(x-t,y)}{2\tan(t/2)}\, dt,$$

$$\tilde{f}^{(0,1)}(x,y) = \text{p.v. } \frac{1}{\pi} \int_{\mathbf{T}} \frac{f(x,y-u)}{2\tan(u/2)}\, du$$

and

$$\tilde{f}^{(1,1)}(x,y) = \text{p.v. } \frac{1}{\pi^2} \int_{\mathbf{T}} \int_{\mathbf{T}} \frac{f(x-t,y-u)}{4\tan(t/2)\tan(u/2)}\, dt\, du.$$

We remark that the analogues of (1)–(4) and the analogues of Theorem A, B and C are true in this case (cf. Weisz [17] and the references there).

7. RIESZ SUMMABILITY OF TWO-PARAMETER FOURIER SERIES

The *Riesz means* of a distribution f are defined by

$$\begin{aligned}\sigma_{n,m}^{\alpha,\beta,\gamma,\delta} f(x,y) :&= \sum_{k=-n}^{n} \sum_{l=-m}^{m} \left(1 - \left|\frac{k}{n+1}\right|^{\gamma}\right)^{\alpha} \left(1 - \left|\frac{l}{m+1}\right|^{\delta}\right)^{\beta} \hat{f}(k,l)\, e^{\imath kx} e^{\imath ly} \\ &=: f * (\kappa_n^{\alpha,\gamma} \times \kappa_m^{\beta,\delta})(x,y)\end{aligned}$$

where

$$\kappa_n^{\alpha,\gamma}(x) := \sum_{k=-n}^{n} \left(1 - \left|\frac{k}{n+1}\right|^{\gamma}\right)^{\alpha} e^{\imath kx} \qquad (x \in \mathbf{T})$$

is the periodic Riesz kernel. Similarly, we introduce the *conjugate Riesz means* of a distribution f by

$$\tilde{\sigma}_{n,m}^{(1,0);\alpha,\beta,\gamma,\delta} f(x,y) := \sum_{k=-n}^{n} \sum_{l=-m}^{m} \left(1-\left|\frac{k}{n+1}\right|^{\gamma}\right)^{\alpha} \left(1-\left|\frac{l}{m+1}\right|^{\delta}\right)^{\beta} (-\imath \operatorname{sign} k)\, \hat{f}(k,l)\, e^{\imath kx} e^{\imath ly}$$
$$= \tilde{f}^{(1,0)} * (\kappa_n^{\alpha,\gamma} \times \kappa_m^{\beta,\delta})(x,y),$$

$$\tilde{\sigma}_{n,m}^{(0,1);\alpha,\beta,\gamma,\delta} f(x,y) := \sum_{k=-n}^{n} \sum_{l=-m}^{m} \left(1-\left|\frac{k}{n+1}\right|^{\gamma}\right)^{\alpha} \left(1-\left|\frac{l}{m+1}\right|^{\delta}\right)^{\beta} (-\imath \operatorname{sign} l)\, \hat{f}(k,l)\, e^{\imath kx} e^{\imath ly}$$
$$= \tilde{f}^{(0,1)} * (\kappa_n^{\alpha,\gamma} \times \kappa_m^{\beta,\delta})(x,y)$$

and

$$\tilde{\sigma}_{n,m}^{(1,1);\alpha,\beta,\gamma,\delta} f(x,y) := \sum_{k=-n}^{n} \sum_{l=-m}^{m} \left(1-\left|\frac{k}{n+1}\right|^{\gamma}\right)^{\alpha} \left(1-\left|\frac{l}{m+1}\right|^{\delta}\right)^{\beta} (-\operatorname{sign}(kl))\, \hat{f}(k,l)\, e^{\imath kx} e^{\imath ly}$$
$$= \tilde{f}^{(1,1)} * (\kappa_n^{\alpha,\gamma} \times \kappa_m^{\beta,\delta})(x,y),$$

respectively. The *maximal* and *maximal conjugate Riesz operators* are defined by

$$\tilde{\sigma}_*^{(i,j);\alpha,\beta,\gamma,\delta} f := \sup_{n,m \in \mathbf{N}} |\tilde{\sigma}_{n,m}^{(i,j);\alpha,\beta,\gamma,\delta} f|,$$

where $\tilde{\sigma}_{n,m}^{(0,0);\alpha,\beta,\gamma,\delta} f := \sigma_{n,m}^{\alpha,\beta,\gamma,\delta} f$ and we define again $\sigma_*^{\alpha,\beta,\gamma,\delta} f := \tilde{\sigma}_*^{(0,0);\alpha,\beta,\gamma,\delta} f$.

We proved in [19] that

$$\kappa_n^{\alpha,\gamma}(x) = \sqrt{2\pi} \sum_{k=-\infty}^{\infty} K_{m+1}^{\alpha,\gamma}(x+2k\pi),$$

where $n \in \mathbf{N}$, $\alpha, \gamma > 0$ and $x \in \mathbf{T}$ (cf. also Butzer and Nessel [3]). From this it follows that the analogues to (5)–(7) hold, namely, for $n \in \mathbf{N}$ and $0 < \alpha \leqslant 1 \leqslant \gamma$,

$$\int_{\mathbf{T}} |\kappa_n^{\alpha,\gamma}|\, d\lambda \leqslant C, \qquad |\kappa_n^{\alpha,\gamma}(x)| \leqslant \frac{C}{n^{\alpha} |x|^{\alpha+1}} \qquad (x \in \mathbf{T}, x \neq 0)$$

and

$$|(\kappa_n^{\alpha,\gamma})'(x)| \leqslant \frac{C}{n^{\alpha-1}|x|^{\alpha+1}} \qquad (x \in \mathbf{T}, x \neq 0).$$

Using these estimates we can prove the following results in the same way as in Sections 4 and 5, so we omit the proofs.

THEOREM 7. *Assume that* $0 < \alpha, \beta \leqslant 1 \leqslant \gamma, \delta$ *and* $1 < p \leqslant \infty$. *Then*

$$\|\sigma_*^{\alpha,\beta,\gamma,\delta} f\|_p \leqslant C_p \|f\|_p \qquad (f \in L_p(\mathbf{T}^2)).$$

COROLLARY 6. *Assume that* $0 < \alpha, \beta \leqslant 1 \leqslant \gamma, \delta$ *and* $1 < p < \infty$. *Then for every* $f \in L_p(\mathbf{T}^2)$ *we have*

$$\sigma_{n,m}^{\alpha,\beta,\gamma,\delta} f \to f \qquad \text{a.e. and in } L_p(\mathbf{T}^2) \text{ norm as} \qquad n, m \to \infty.$$

THEOREM 8. *Assume that* $0 < \alpha, \beta \leqslant 1 \leqslant \gamma, \delta$ *and* $i, j = 0, 1$. *Then*

$$\|\tilde{\sigma}_*^{(i,j);\alpha,\beta,\gamma,\delta} f\|_{p,q} \leqslant C_{p,q} \|f\|_{H_{p,q}(\mathbf{T}\times\mathbf{T})} \qquad (f \in H_{p,q}(\mathbf{T}\times\mathbf{T}))$$

for every $\max\{1/(\alpha+1), 1/(\beta+1)\} < p < \infty$ *and* $0 < q \leqslant \infty$. *Especially, if* $f \in H_1^\sharp(\mathbf{T}\times\mathbf{T})$ *then*

$$\lambda(\tilde{\sigma}_*^{(i,j);\alpha,\beta,\gamma,\delta} f > \rho) \leqslant \frac{C}{\rho} \|f\|_{H_1^\sharp(\mathbf{T}\times\mathbf{T})} \qquad (\rho > 0).$$

Note that a very special case of Theorem 8, i.e., if $i = j = 0$, $\alpha = \beta = \gamma = \delta = 1$ was proved in Weisz [17] only for $3/4 < p < \infty$.

COROLLARY 7. *Assume that* $0 < \alpha, \beta \leqslant 1 \leqslant \gamma, \delta$ *and* $i, j = 0, 1$. *If* $f \in H_1^\sharp(\mathbf{T}\times\mathbf{T})$ $(\supset L \log L(\mathbf{T}^2))$ *then*

$$\tilde{\sigma}_{n,m}^{(i,j);\alpha,\beta,\gamma,\delta} f \to \tilde{f}^{(i,j)} \qquad \text{a.e.} \quad \text{as} \quad n, m \to \infty.$$

THEOREM 9. *Assume that* $0 < \alpha, \beta \leqslant 1 \leqslant \gamma, \delta$, $i, j = 0, 1$ *and* $n, m \in \mathbf{N}$. *Then*

$$\|\tilde{\sigma}_{n,m}^{(i,j);\alpha,\beta,\gamma,\delta} f\|_{H_{p,q}(\mathbf{T}\times\mathbf{T})} \leqslant C_{p,q} \|f\|_{H_{p,q}(\mathbf{T}\times\mathbf{T})} \qquad (f \in H_{p,q}(\mathbf{T}\times\mathbf{T}))$$

for every $\max\{1/(\alpha+1), 1/(\beta+1)\} < p < \infty$ *and* $0 < q \leqslant \infty$.

COROLLARY 8. *Assume that* $0 < \alpha, \beta \leqslant 1 \leqslant \gamma, \delta$ *and* $i, j = 0, 1$. *If* $\max\{1/(\alpha+1), 1/(\beta+1)\} < p < \infty$, $0 < q \leqslant \infty$ *and* $f \in H_{p,q}(\mathbf{T} \times \mathbf{T})$ *then*

$$\tilde{\sigma}_{n,m}^{(i,j);\,\alpha,\beta,\gamma,\delta} f \to \tilde{f}^{(i,j)} \qquad \textit{in } H_{p,q}(\mathbf{T} \times \mathbf{T}) \textit{ norm} \quad \textit{as} \quad n, m \to \infty.$$

REFERENCES

1. C. Bennett and R. Sharpley, Interpolation of operators, *in* "Pure and Applied Mathematics," Vol. 129. Academic Press, New York, 1988.
2. J. Bergh and J. Löfström, "Interpolation Spaces, an Introduction," Springer-Verlag, Berlin/Heidelberg, New York, 1976.
3. P. L. Butzer and R. J. Nessel, "Fourier Analysis and Approximation," Birkhäuser Verlag, Basel, 1971.
4. S.-Y. A. Chang and R. Fefferman, Some recent developments in Fourier analysis an H^p-theory on product domains, *Bull. Amer. Math. Soc.* **12** (1985), 1–43.
5. P. Duren, "Theory of H^p Spaces," Academic Press, New York, 1970.
6. R. E. Edwards, "Fourier Series, a Modern Introduction," Vol. 2, Springer-Verlag, Berlin/Heidelberg/New York, 1982.
7. C. Fefferman and E. M. Stein, H^p spaces of several variables, *Acta Math.* **129** (1972), 137–194.
8. R. Fefferman, Calderon–Zygmund theory for product domains: H^p spaces, *Proc. Nat. Acad. Sci. U.S.A.* **83** (1986), 840–843.
9. A. P. Frazier, The dual space of H^p of the polydisc for $0 < p < 1$, *Duke Math. J.* **39** (1972), 369–379.
10. R. F. Gundy, Maximal function characterization of H^p for the bidisc, *in* "Lecture Notes in Math," Vol. 781, pp. 51–58, Springer-Verlag, Berlin/Heidelberg/New York, 1982.
11. R. F. Gundy and E. M. Stein, H^p theory for the poly-disc, *Proc. Nat. Acad. Sci. U.S.A.* **76** (1979), 1026–1029.
12. B. S. Kashin and A. A. Saakjan, "Orthogonal Series," Translations of Mathematical Monographs, Vol. 75, Amer. Math. Soc., Providence, 1989.
13. K.-C. Lin, Interpolation between Hardy spaces on the bidisc, *Studia Math.* **84** (1986), 89–96.
14. J. Marcinkievicz and A. Zygmund, On the summability of double Fourier series, *Fund. Math.* **32** (1939), 122–132.
15. F. Schipp, W. R. Wade, P. Simon, and J. Pál, "Walsh Series: an Introduction to Dyadic Harmonic Analysis," Hilger, Bristol/New York, 1990.
16. E. M. Stein and G. Weiss, "Introduction to Fourier Analysis on Euclidean Spaces," Princeton Univ. Press, Princeton, NJ, 1971.
17. F. Weisz, Cesàro summability of two-parameter trigonometric-Fourier series, *J. Approx. Theory*. **90** (1997), 30–45.
18. F. Weisz, Martingale Hardy spaces and their applications in Fourier-analysis, *in* "Lecture Notes in Math.," Vol. 1568, Springer-Verlag, Berlin/Heidelberg/New York, 1994.
19. F. Weisz, Riesz means of Fourier transforms and Fourier series on Hardy spaces, *Studia Math.* **131** (1998), 253–270.
20. N. Wiener, "The Fourier Integral and Certain of Its Applications," Dover, New York, 1959.
21. A. Zygmund, "Trigonometric Series," Cambridge Univ. Press, London, 1959.